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Almost-Sure Asymptotic Stability of a General Four-Dimensional System Driven by Real Noise

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In the first part of this paper, we construct an asymptotic expansion for the maximal Lyapunov exponent, the exponential growth rate of solutions to a linear stochastic system, and the rotation numbers for a general four-dimensional dynamical system driven by a small-intensity real noise process. Stability boundaries are obtained provided the natural frequencies are noncommensurable and the infinitesimal generator associated with the noise process has an isolated simple zero eigenvalue. This work is an extension of the work of Sri Namachchivaya and Van Roessel and is general in the sense that general stochastic perturbations of nonautonomous systems with two noncommensurable natural frequencies are considered.

KEY WORDS: Lyapunov exponents; rotation numbers; Itô equations; almost-sure asymptotic stability.

1. INTRODUCTION

One of the primary concerns in the analysis of dynamical systems is the determination of the stability of the steady-state solutions. This analysis becomes more difficult when these systems are excited by a stochastic process. The stability of a linear stochastic system can be defined in several ways. The weakest, or least conservative, definition is that of stability in distribution. A more conservative estimate of the stability boundary is described by stability in probability. Thus, if a system is stable in probability, it is also stable in distribution. The last two definitions of stability in the stochastic sense are stability in the rth mean and almost-sure stability, or stability with probability one. If a dynamical system excited by noise is stable according to either of these definitions, it is stable in distribution and

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in probability, as well. However, rth mean and almost-sure stability do not imply each other, i.e., a system can be almost surely stable while its second moments grow exponentially. Kozin and Sugimoto,[10] with extensions by Arnold,[2] established a characterization between moment stability and almost-sure stability for linear Itô stochastic differential equations when the process is ergodic on the entire surface of the n-sphere. It was shown that the region of sample stability is the limit of the region of rth moment stability for r approaching zero.

Stability in the almost-sure sense is determined by the sign of the maximal Lyapunov exponent. It was shown by Arnold and Kliemann[3] that, for a linear system with stochastic parametric excitation, the Lyapunov exponents are analogous to the real part of the eigenvalue. Thus, the maximal Lyapunov exponent yields the almost-sure asymptotic stability of the linear system. We can also define the stochastic analog of the imaginary part of the eigenvalue; the rotation number determines the asymptotic rate of rotation for the stochastically perturbed system.

In the present analysis, we approximate the maximal Lyapunov exponent and rotation number for a general four-dimensional linear system excited by noise. For the case when the noise is white, Khas'minskii[8] presented necessary and sufficient conditions under which the system is stable with probability one without explicit mention of the Lyapunov exponent. The studies by Kozin and Prodromou[9] and Mitchell and Kozin[11] yielded results for second-order systems and a complete examination by Nishioka[12] considered the effects of all possible singularities that may be present in a one-dimensional diffusion process.

In the case of ergodic but nonwhite noise excitation, few results are available. The existing results are due to Arnold *et al.*[4] and Pardoux and Wihstutz.[14] A survey paper of 1991 by Pinsky and Wihstutz[15] summarizes the previous work on this topic. A more recent investigation was performed by Sri Namachchivaya[16] in which the almost-sure stability of dynamical systems under the combined influence of stochastic and harmonic excitation was examined.

As in most of the studies involving multi-degree-of-freedom systems reported to date, the analytical results in ref. 16 were derived under the condition that only one mode is critical while the remaining modes are strongly stable. This, however, is not necessarily true in all physical systems. For this reason, it is imperative to determine the almost-sure asymptotic stability of multi-degree-of-freedom dynamical systems with more than one critical mode. The maximal Lyapunov exponent and rotation number for stochastically perturbed codimension-two bifurcations have been calculated via the method of averaging by Sri Namachchivaya and Talwar.[17] In ref. 17, averaging was applied to obtain a set of approximate Itô equations for amplitudes and phases. However in order to decouple the amplitude and phase equations completely, certain restrictive conditions on the manner in which the noise entered the equations were imposed.

The focus of this work is to approximate the maximal Lyapunov exponent for a four-dimensional system with two critical modes perturbed by a small-intensity multiplicative real noise process. The approach adopted here is the perturbation method developed by Sri Namachchivaya and Van Roessel.[18] Using this approach, no restrictions on the structure of the stochastic terms in the equations of motion are necessary to decouple the amplitude and phase equations. Thus, the results presented here are for a general four-dimensional system parametrically perturbed by a real noise process. As in ref. 18, the frequencies are noncommensurable and the infinitesimal generator associated with the noise process is assumed to have an isolated simple zero eigenvalue.

Section 2 describes the formulation of the mathematical problem. General results for the probability density for all possible singular cases are presented in Section 3 and the maximal Lyapunov exponent is evaluated in Section 4 and the rotation number for each case in Section 5. Section 6 summarizes the contributions of this research.

2. STATEMENT OF THE PROBLEM AND FORMULATION

Consider a linear stochastic system governed by the following equations of motion:

$$\dot{x} = Ax - \varepsilon f(\xi(t))\, Bx, \qquad x \in R^4 \tag{1}$$

Appropriate scaling of the matrix A yields

$$A = A_0 - \varepsilon^2 A_1$$

where

$$A_0 = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_2 & 0 \\ 0 & 0 & 0 & \delta_2 \end{bmatrix}$$

The matrix B is described by $B = [b_{ij}]$ and the quantities δ_1 and δ_2 are damping parameters. In this analysis, it is assumed that the frequencies ω_1 and ω_2 are noncommensurable. The term $\xi(t)$ is a small-intensity real noise process defined on a smooth connected one-dimensional Riemannian manifold M (with or without boundary). The smooth function $f: M \to R$ is assumed to have zero mean.

Before proceeding, a brief description of some of the results of Oseledec's Multiplicative Ergodic Theorem,[13] as related to four-dimensional systems, is necessary. Consider the linear stochastic system in Eq. (1) under the assumption that $\xi(t)$ is ergodic. According to the Multiplicative Ergodic Theorem, the Lyapunov exponent of the solution of Eq. (1), $x(t; x_0)$, for the initial condition x_0 ($x_0 \neq 0$) is

$$\lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \log \|x(t; x_0)\| \tag{2}$$

where $\lambda(x_0)$ takes on one of r fixed or nonrandom values $\lambda_1 < \cdots < \lambda_r$. Which λ_i is realized depends on the initial condition x_0. The multiplicities of the Lyapunov exponents sum to the dimension of the system n (in this case, $n = 4$). Associated with each λ_i there exists a random linear invariant subspace E_i, known as an Oseledec space, such that $E_1 \oplus E_2 \oplus \cdots \oplus E_r = \mathbf{R}^n$, with

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|x(t; x_0)\| = \lambda_i \qquad \text{iff} \quad x(t; x_0) \in E_i$$

The dimension of each Oseledec space E_i is given by the multiplicity of the associated Lyapunov exponent λ_i.

The effects of deterministic detuning and noise on the Lyapunov exponents of the system under consideration are depicted in Fig. 1. In the absence of noise and detuning, all eigenvalues lie on the imaginary axis. The addition of negative detuning stabilizes the system. In this case, as shown, all Lyapunov exponents are negative. In the presence of both deterministic detuning and noise, generically one expects four distinct Lyapunov exponents. The maximum of these, λ_1, determines the almost-sure stability of the stochastic system.

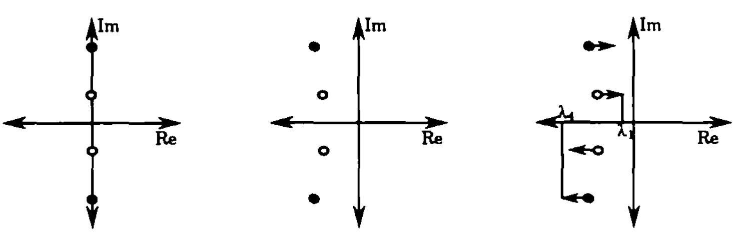

Fig. 1. Effect of detuning and noise on system stability. (a) Zero noise and detuning, (b) detuning only, (c) noise and detuning.

The stability of the system described by Eq. (1) was studied by Sri Namachchivaya and Talwar[17] using the method of stochastic averaging to derive a set of approximate Itô equations for the amplitudes and phases. Using that method, it is necessary to impose certain restrictive conditions on the matrix B in order to decouple the amplitude and phase equations completely. Employing the notation

$$H_{ij}^{\pm} = b_{2i-1,2j} \pm b_{2i,2j-1} \qquad \text{and} \qquad J_{ij}^{\pm} = b_{2i,2j} \pm b_{2i-1,2j-1}$$

these restrictions are given as

$$H_{12}^{-} J_{21}^{+} + H_{21}^{-} J_{12}^{+} = 0 \qquad \text{and} \qquad H_{21}^{+} J_{12}^{-} - H_{12}^{+} J_{21}^{-} = 0 \tag{3}$$

and either

$$H_{11}^{-} = H_{22}^{-} = 0 \qquad \text{or} \qquad J_{11}^{+} = J_{22}^{+} = 0 \tag{4}$$

Examples of systems in which these conditions are satisfied include the double oscillator described by Ariaratnam and Xie[1] and problems with a symmetric B matrix. These conditions are also satisfied by the system considered by Sri Namachchivaya and Van Roessel.[18] In the present analysis, no restrictions on the structure of the B matrix are required to decouple the amplitude and phase equations. Thus, the results obtained here are for the most general case of Eq. (1). The analysis presented in this paper is based heavily on the work completed by Sri Namachchivaya and Van Roessel.[18]

Let G denote the infinitesimal generator of $\xi(t)$, i.e.,

$$G(\xi) = \mu_i(\xi) \frac{\partial}{\partial \xi_i} + \frac{1}{2} \sigma_{ij}^2(\xi) \frac{\partial^2}{\partial \xi_i \, \partial \xi_j}$$

and, as in ref. 18, assume that G has an isolated simple zero eigenvalue. This implies that $u \equiv \text{const}$ is the only solution of $Gu = 0$. Consequently, the adjoint operator G^* must also have an isolated simple zero eigenvalue. The results obtained from this analysis are applicable when the manifold M is of arbitrary dimension. For the case in which M is one-dimensional, the normalized invariant measure $v(\xi)\, d\xi$ satisfying the Fokker–Planck equation $G^* v(\xi) = 0$ can be written in terms of scale measure and speed density as

$$v(\xi) = m(\xi)[c_1 S(\xi) + c_2]$$

where

$$S(\xi) = \int^{\xi} s(\eta)\, d\eta, \qquad m(\xi) = [\sigma^2(\xi)\, s(\xi)]^{-1}$$

$$s(\xi) = \exp\left\{ -\int^{\xi} \frac{2\mu(\eta)}{\sigma^2(\eta)}\, d\eta \right\}$$

The constants c_1 and c_2 are determined using boundary and normality conditions, respectively.

By the usual transformation, i.e.,

$$x_{2i-1} = r_i \cos\phi_i, \qquad x_{2i} = r_i \sin\phi_i, \qquad \rho_i = \ln(r_i)$$

we obtain amplitude and phase equations of the form

$$\dot{\rho}_i = \varepsilon^2[\tilde{p}_i(\phi)] + \varepsilon[p_i(e^{(\rho_j - \rho_i)}, \phi)]\, f(\xi(t)) \tag{5}$$

$$\dot{\phi}_i = [\omega_i + \varepsilon^2 \tilde{h}_i(\phi)] + \varepsilon[h_i(e^{(\rho_j - \rho_i)}, \phi)]\, f(\xi(t)) \tag{6}$$

In this form, the amplitude and phase equations are coupled by the presence of terms of the form $e^{(\rho_j - \rho_i)}$ for $i \neq j$. Since $e^{(\rho_j - \rho_i)}$ is always positive, we can introduce a one-to-one mappings $e^{(\rho_j - \rho_i)} = \tan\theta$, $\theta \in [0, \pi/2]$. Thus, applying the transformations

$$x_1 = e^{\rho} \cos\phi_1 \cos\theta, \qquad x_2 = -e^{\rho} \sin\phi_1 \cos\theta$$

$$x_3 = e^{\rho} \cos\phi_2 \sin\theta, \qquad x_4 = -e^{\rho} \sin\phi_2 \sin\theta$$

to the original system yields the following set of equations for the amplitude ρ, phase variables (ϕ_1, ϕ_2, θ), and noise process ξ:

$$\dot{\rho} = \varepsilon f(\xi)\, q_1(\phi_1, \phi_2, \theta) + \varepsilon^2 \tilde{q}_1(\theta) \tag{7}$$

$$\dot{\phi}_i = \omega_i + \varepsilon f(\xi)\, h_i(\phi_1, \phi_2, \theta) + \varepsilon^2 \tilde{h}_i(\phi_1, \phi_2, \theta) \tag{8}$$

$$\dot{\theta} = \varepsilon f(\xi)\, q_2(\phi_1, \phi_2, \theta) + \varepsilon^2 \tilde{q}_2(\theta) \tag{9}$$

$$d\xi = \mu(\xi)\, dt + \sigma(\xi) \circ dW_t \tag{10}$$

where the expressions for q_i, $\tilde{q}_i$, and h_i $(i = 1, 2)$ are given explicitly in the appendix. Given the structure of the A_1 matrix it can be shown that $\tilde{h}_i \equiv 0$ for $i = 1\ 2$.

The processes $(\phi_1, \phi_2, \theta, \xi)$ are independent of the amplitude ρ and form a diffusive Markov process with associated generator

$$L^{\varepsilon} = L^0 + \varepsilon L^1 + \varepsilon^2 L^2$$

where, in general,

$$L^0 = \sum_{i=1}^{2} \omega_i \frac{\partial}{\partial \phi_i} + G(\xi)$$

$$L^1 = f(\xi) \left[q_2 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2} h_i \frac{\partial}{\partial \phi_i} \right]$$

$$L^2 = \tilde{q}_2 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2} \tilde{h}_i \frac{\partial}{\partial \phi_i}$$

For this particular A_1 matrix, $L^2 = \tilde{q}_2\, \partial/\partial\theta$. We also define the function $Q^\varepsilon(\phi_1, \phi_2, \theta, \xi)$ such that

$$Q^\varepsilon = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2$$

In the present analysis, we can write Q^ε in terms of $f(\xi)$, q_1, and $\tilde{q}_1$ as follows:

$$Q^\varepsilon(\phi_1, \phi_2, \theta, \xi) = \varepsilon f(\xi)\, q_1(\phi_1, \phi_2, \theta) + \varepsilon^2 \tilde{q}_1(\theta)$$

Then, according to Oseledec's Multiplicative Ergodic Theorem,[13] assuming the operator L^ε to be ergodic, the maximal Lyapunov exponent is given by

$$\lambda^\varepsilon = \langle Q^\varepsilon, p^\varepsilon \rangle = \int_0^{\pi/2} \int_M \int_0^{2\pi} \int_0^{2\pi} Q^\varepsilon p^\varepsilon \, d\phi_1 \, d\phi_2 \, d\xi \, d\theta$$

where p^ε is the unique ergodic invariant measure associated with the generator L^ε, i.e., p^ε solves the Fokker–Planck equation given by

$$L^{\varepsilon *} p^\varepsilon = 0$$

provided L^ε is hypoelliptic.

We construct a formal expansion of the invariant measure, i.e.,

$$p^\varepsilon = p^0 + \varepsilon p^1 + \cdots + \varepsilon^N p^N + \cdots$$

Substituting this expansion and the expansion for L^ε into the Fokker–Planck equation yields the following sequence of Poisson equations to be solved for p^0, p^1, p^2,...:

$$\begin{aligned} L^{0*} p^0 &= 0 \\ L^{0*} p^1 &= -L^{1*} p^0 \\ L^{0*} p^2 &= -L^{1*} p^1 - L^{2*} p^0 \end{aligned} \tag{11}$$

This yields the following expression for the maximal Lyapunov exponent:

$$\begin{aligned} \chi^\varepsilon = \langle Q^0, p^0 \rangle &+ \varepsilon[\langle Q^1, p^0 \rangle + \{Q^0, p^1 \rangle] \\ &+ \varepsilon^2[\langle Q^2, p^0 \rangle + \langle Q^1, p^1 \rangle + \langle Q^0, p^2 \rangle] + \cdots \end{aligned}$$

A proof that this expansion is, in fact, asymptotic begins with the construction of the adjoint problem $L^\varepsilon F^\varepsilon = Q^\varepsilon$ with $F^\varepsilon = F^0 + \varepsilon F^1 + \cdots + \varepsilon^N F^N$ as in ref. 4. In this expression, $F^0, F^1, \ldots, F^N$ are such that

$$\begin{aligned}(L^0 + \varepsilon^1 + \varepsilon^2 L^2)&(F^0 + \varepsilon F^1 + \cdots + \varepsilon^N F^N)\\ &= Q^\varepsilon - (q^0 + \varepsilon q^1 + \cdots + \varepsilon^N q^N)\\ &\quad + \varepsilon^{N+1}\{L^1 F^N + L^2 F^{N-1}\} + \varepsilon^{N+2}\{L^2 F^N\}\end{aligned}$$

The functions $q^0, q^1, \ldots, q^N$ are independent of θ, ϕ_1, and ϕ_2 and satisfy the equations

$$\begin{aligned}L^0 F^0 &= Q^0 - q^0\\ L^0 F^1 &= Q^1 - q^1 - L^1 F^0\\ L^0 F^2 &= Q^2 - q^2 - L^1 F^1 - L^2 F^0\\ &\vdots\\ L^0 F^N &= -q^N - L^1 F^{N-1} - L^2 F^{N-2}\end{aligned}$$

Next we define the truncated density $\tilde{p}^\varepsilon = p^0 + \varepsilon p^1 + \cdots + \varepsilon^N p^N$ and assume $v(\xi)$ is the marginal of both p^ε and $\tilde{p}^\varepsilon$ on M. Employing Eqs. (11), the error introduced by truncating λ^ε at an arbitrary order $N \geqslant 0$ is given by

$$\begin{aligned}\langle Q^\varepsilon, p^\varepsilon\rangle &- \langle Q^\varepsilon, \tilde{p}^\varepsilon\rangle\\ &= -\varepsilon^{N+1}[\langle L^1 F^N + L^2 F^{N-1}, p^\varepsilon - \tilde{p}^\varepsilon\rangle + \langle L^{1*} p^N + L^{2*} p^{N-1}, F^\varepsilon\rangle\\ &\quad - \langle Q^1, p^N\rangle - \langle Q^2, p^{N-1}\rangle]\\ &\quad - \varepsilon^{N+2}[\langle L^2 F^N, p^\varepsilon - \tilde{p}^\varepsilon\rangle + \langle L^{2*} p^N, F^\varepsilon\rangle - \langle Q^2, p^N\rangle]\end{aligned}$$

Suppose that the functions $p^0, p^1, \ldots, p^N$ and $F^0, F^1, \ldots, F^N$ are such that all inner products above are well defined. Since p^ε is unknown, we can assume that $p^0, p^1, \ldots, p^N$ are constructed such that

$$\sup_{\phi,\theta,\xi} |L^1 F^N + L^2 F^{N-1}| \leqslant K_1 < \infty \qquad \text{and} \qquad \sup_{\phi,\theta,\xi} |L^2 F^N| \leqslant K_2 < \infty$$

Applying the above estimate, it is clear that the expansion for a fixed $N \geqslant 0$ is a valid asymptotic expansion.

Since $Q^0 = 0$, the expression for the maximal Lyapunov exponent reduces to

$$\lambda^\varepsilon = \varepsilon\langle Q^1, p^0\rangle + \varepsilon^2[\langle Q^2, p^0\rangle + \langle Q^1, p^1\rangle] + \cdots$$

where p^0 and p^1 satisfy the Poisson equations above with the periodic boundary conditions

$$p^0(\phi_1, \phi_2, \theta, \xi) = p^0(\phi_1 + 2\pi, \phi_2, \theta, \xi) = p^0(\phi_1, \phi_2 + 2\pi, \theta, \xi)$$
$$p^1(\phi_1, \phi_2, \theta, \xi) = p^1(\phi_1 + 2\pi, \phi_2, \theta, \xi) = p^1(\phi_1, \phi_2 + 2\pi, \theta, \xi)$$

Solving the $O(1)$ Poisson equation with appropriate boundary conditions and considering the noncommensurability condition on the natural frequencies yields

$$p^0(\xi, \theta) = \frac{v(\xi)\, F(\theta)}{4\pi^2}$$

where $v(\xi)$ is the invariant measure satisfying $G^* v = 0$. Note that for arbitrary $F(\theta)$, the inner product $\langle Q^1, p^0 \rangle = 0$ due to the periodic boundary conditions on ϕ_1 and ϕ_2 *and* the zero mean assumption on $f(\xi)$. The maximal Lyapunov exponent, up to $O(\varepsilon^2)$, reduces to

$$\lambda^{\varepsilon} = \varepsilon^2 [\langle Q^2, p^0 \rangle + \langle Q^1, p^1 \rangle] \tag{12}$$

The $O(\varepsilon)$ Poisson equation and its adjoint are

$$L^{0*} p^1 = -L^{1*} p^0 \qquad \text{and} \qquad L^0 u = 0$$

and the associated solvability condition is

$$\langle L^{1*} p^0, u \rangle = 0 \qquad \forall u \in \ker(L^0) \tag{13}$$

Due to the assumption on G and the boundary conditions on ϕ_1 and ϕ_2 we have

$$\ker(L^0) = \{C(\theta): C \text{ is an arbitrary function of } \theta\}$$

and $\ker(L^0)$ is one-dimensional by the assumption on the generator G. Thus, the solution for p^1 exists and is unique.

The solvability condition of Eq. (13) reduces to

$$\int_0^{\pi/2} \tilde{F}(\theta)\, C(\theta)\, d\theta = 0$$

where

$$\tilde{F}(\theta) = \int_M f(\xi)\, v(\xi) \int_0^{2\pi} \int_0^{2\pi} \left[\frac{\partial}{\partial \theta}(q_2 F) + F \sum_{i=1}^{2} \frac{\partial h_i}{\partial \phi_i} \right] d\phi_1\, d\phi_2\, d\xi$$

which must hold for arbitrary $C(\theta)$. This implies

$$\tilde{F}(\theta)=0$$

This condition is automatically satisfied due to the periodicity of ϕ_1 and ϕ_2. Thus, we are not able to determine $F(\theta)$ using the $O(\varepsilon)$ solvability condition. We can, however, find an expression for p^1 in terms of $F(\theta)$. To do so, rewrite the $O(\varepsilon)$ Poisson equation as

$$\left(G^* - \omega_i \sum_{i=1}^{2} \frac{\partial}{\partial \phi_i}\right) p^1 = -\frac{f(\xi)\, v(\xi)}{4\pi^2} R(\phi_1, \phi_2, \theta) \tag{14}$$

where

$$\begin{aligned}
R(\phi_1, \phi_2, \theta)&\\
= -\frac{1}{2}(J_{22}^+ - J_{11}^+)&\left[(c_\theta^2 - s_\theta^2)F + \frac{1}{2}s_{2\theta}\frac{\partial F}{\partial \theta}\right]\\
&-\left[\left(1+\frac{1}{2}c_{2\theta}\right)F + \frac{1}{4}s_{2\theta}\frac{\partial F}{\partial \theta}\right](J_{11}^- c_{2\phi_1} + H_{11}^+ s_{2\phi_1})\\
&-\left[\left(1-\frac{1}{2}c^{2\theta}\right)F - \frac{1}{4}s_{2\theta}\frac{\partial F}{\partial \theta}\right](J_{22}^- c_{2\phi_2} + H_{22}^+ s_{2\phi_2})\\
&+\frac{1}{2}\left[(2c_\theta s_\theta + t_\theta)F + s_\theta^2\frac{\partial F}{\partial \theta}\right](J_{12}^+ C^- - J_{12}^- C^+ + H_{12}^- S^- - H_{12}^+ S^+)\\
&+\frac{1}{2}\left[\left(2c_\theta s_\theta + \frac{1}{t_\theta}\right)F - c_\theta^2\frac{\partial F}{\partial \theta}\right](J_{21}^+ C^- - J_{21}^- C^+ - H_{21}^- S^- - H_{21}^+ S^+)
\end{aligned}$$

In the above expression, we have used $c_{(\cdot)}=\cos(\cdot)$, $s_{(\cdot)}=\sin(\cdot)$, $t_{(\cdot)}=\tan(\cdot)$, and $C^{\pm}=\cos(\omega_1 \pm \omega_2)$, $S^{\pm}=\sin(\omega_1 \pm \omega_2)$.

Introduce an auxiliary time t such that Eq. (14) becomes

$$\left(\frac{\partial}{\partial t} - G^* + \omega_i \sum_{i=1}^{2} \frac{\partial}{\partial \phi_i}\right) p_t^1 = \frac{f(\xi)\, v(\xi)}{4\pi^2} R(\phi_1, \phi_2, \theta) \tag{15}$$

The density p^1 is the stationary solution of Eq. (15) and solves Eq. (14), i.e.,

$$p^1(\phi_1, \phi_2, \theta, \xi) = \lim_{t\to\infty} [p_t^1(\phi_1, \phi_2, \theta, \xi, t)]$$

Employing the transformation

$$\tau = \frac{1}{2}\left[t + \frac{1}{2}\left(\frac{\phi_1}{\omega_1} + \frac{\phi_2}{\omega_2}\right)\right], \qquad s = \frac{1}{2}\left[t - \frac{1}{2}\left(\frac{\phi_1}{\omega_1} + \frac{\phi_2}{\omega_2}\right)\right]$$

and $\Upsilon=\omega_1\phi_2-\omega_2\phi_1$ in Eq. (15) yields

$$\left(\frac{\partial}{\partial\tau}-G^*\right)p_t^1=\frac{H(\xi)}{4\pi^2}R(\phi_1(\tau,s,\Upsilon),\phi_2(\tau,s,\Upsilon),\theta) \tag{16}$$

where $H(\xi)=f(\xi)\,v(\xi)$. Equation (16) is an inhomogeneous boundary value problem. We can transform this into a homogeneous initial value problem using Duhamel's principle with zero initial conditions (see, for example, ref. 19). The solution to Eq. (16) can then be written as

$$p_t^1(\tau,s,\Upsilon,\xi)=\frac{1}{4\pi^2}\int_0^\tau R(\phi_1(\tau-T,s,\Upsilon),\phi_2(\tau-T,s,\Upsilon),\theta)\,K(\xi,T)\,dT$$

where $g(\xi,T;\eta,0)$ is the transient density which solves

$$\frac{\partial g}{\partial t}=G^*g,\qquad g(\xi,0;\eta,0)=\delta(\xi-\eta)$$

and

$$K(\xi,T)=\int_M H(\eta)\,g(\xi,T;\eta,0)\,d\eta$$

The final form of $p^1(\phi_1,\phi_2,\theta,\xi)$ is found by taking the limit as $\tau\to\infty$:

$$p^1(\phi_1,\phi_2,\theta,\xi)=\frac{1}{4\pi^2}\int_0^\infty R(\omega_1T-\phi_1,\omega_2T-\phi_2,\theta,T)\,K(\xi,T)\,dT \tag{17}$$

In Eq. (17), $R(\omega_1T-\phi_1,\ \omega_2T-\phi_2,\ \theta,\ T)$ contains $F(\theta)$ and its derivatives which have yet to be determined. This can be accomplished with the aid of the $O(\varepsilon^2)$ solvability condition. Recall the $O(\varepsilon^2)$ Poisson equation

$$\left(G^*-\omega_i\sum_{i=1}^{2}\frac{\partial}{\partial\phi_i}\right)p^2=\chi_0(\phi_1,\phi_2,\theta,\xi)+\chi_1(\phi_1,\phi_2,\theta,\xi) \tag{18}$$

where

$$\chi_0=\frac{\partial}{\partial\theta}(\tilde{q}_2p^0)\qquad\text{and}\qquad\chi_1=f(\xi)\left[\frac{\partial}{\partial\theta}(q_2p^1)+\sum_{i=1}^{2}\frac{\partial}{\partial\phi_i}(h_ip^1)\right]$$

Then we have

$$L^{0*}p^2=\chi_0+\chi_1$$

and the corresponding solvability condition

$$\langle \chi_0 + \chi_1, C(\theta) \rangle = 0 \qquad \forall C(\theta) \in \ker(L^0)$$

Evaluating the solvability condition for arbitrary $C(\theta)$ yields the following ordinary differential equaiton for $F(\theta)$:

$$-\frac{d}{d\theta}\left[\Phi(\theta)\, F(\theta)\right] + \frac{1}{2}\frac{d^2}{d\theta^2}\left[\Psi^2(\theta)\, F(\theta)\right] = 0 \tag{19}$$

where

$$\Psi^2(\theta) = A\cos^2 2\theta + B\cos 2\theta + C$$
$$\Phi(\theta) = -\tfrac{1}{2}(\tilde{\lambda}_1 - \tilde{\lambda}_2)\sin 2\theta + \Psi^2(\theta)\cot 2\theta$$

This is indeed the diffusion equation in θ. This describes the stochastic coupling between ρ_1 and ρ_2 in Eq. (5).

Throughout the remainder of this paper, the following notation will be used:

$$\alpha_{ij} = \tfrac{1}{8}\{[(H_{ij}^-)^2 + (J_{ij}^+)^2]\, S(\Omega^-) + [(H_{ij}^+)^2 + (J_{ij}^-)^2]\, S(\Omega^+)\}$$
$$\beta_i = \tfrac{1}{8}[(H_{ii}^+)^2 + (J_{ii}^-)^2]\, S(2\omega_i)$$
$$\mu = \tfrac{1}{8}[(H_{12}^+ H_{21}^+ + J_{12}^- J_{21}^-)\, S(\Omega^+) - (H_{12}^- H_{21}^- - J_{12}^+ J_{21}^+)\, S(\Omega^-)]$$
$$\gamma_1 = \tfrac{1}{4}(H_{21}^- J_{12}^+ + H_{12}^- J_{21}^+)\, \Gamma(\Omega^-)$$
$$\gamma_2 = \tfrac{1}{4}(H_{21}^+ J_{12}^- - H_{12}^+ J_{21}^-)\, \Gamma(\Omega^+)$$
$$\gamma = \tfrac{1}{16}(J_{11}^+ - J_{22}^+)^2\, S(0) + \tfrac{1}{4}(\beta_1 + \beta_2) - \tfrac{1}{4}(\alpha_{12} + \alpha_{21}) - \tfrac{1}{2}\mu$$

The sine and cosine spectrums are defined, respectively, as

$$\Gamma(\omega) = 2\int_0^\infty R(T)\sin \omega T\, dT, \qquad S(\omega) = 2\int_0^\infty R(T)\cos \omega T\, dT$$

where $R(T)$ is the autocorrelation of $f(\xi)$, i.e.,

$$R(T) = \int_M f(\xi)\, K(\xi, T)\, d\xi$$

and $\Omega^\pm = \omega_1 \pm \omega_2$. Employing this notation, we can write

$$A = -\gamma$$
$$B = -\tfrac{1}{2}(\alpha_{12} - \alpha_{21}) = -\tfrac{1}{2}\alpha^-$$
$$C = \gamma + \tfrac{1}{2}(\alpha_{12} + \alpha_{21}) = \gamma + \tfrac{1}{2}\alpha^+$$
$$\lambda_i = -\delta_i + \beta_i \qquad \text{and} \qquad \tilde{\lambda}_i = \lambda_i + \gamma_i$$

where λ_1 and λ_2 are the Lyapunov exponents for the case when the modes in Eq. (1) are decoupled.

Equation (19) can be rewritten as

$$\frac{d}{d\theta}\left\{-\Phi(\theta)\,F(\theta)+\frac{1}{2}\frac{d}{d\theta}\,[\Psi^2(\theta)\,F(\theta)]\right\}=0 \tag{20}$$

Let $A_{st}(\theta)$ be the term in the brackets. Then, $A_{st}(\theta)$ must be constant with respect to θ, i.e., $A_{st}(\theta)=A_{st}$. By examining Eq. (20), we can see that there may be singularities in the open interval $(0, \pi/2)$. All possible singular cases must be considered when attempting to solve this expression for $F(\theta)$. The location of the singular points and the behavior of the diffusion process in the presence of these singularities will be examined in the next section.

3. EVALUATION OF SOLUTIONS

The solution to the Fokker–Planck equation can be written as

$$F(\theta)=m(\theta)[2A_{st}S(\theta)+c] \tag{21}$$

where the scale and speed measures are defined in terms of Φ and Ψ as

$$m(\theta)=[\Psi^2(\theta)\,s(\theta)]^{-1}$$

$$s(\theta)=\exp\left\{-\int^{\theta}\frac{2\Phi(\eta)}{\Psi^2(\eta)}\,d\eta\right\},\qquad S(\theta)=\int^{\theta}s(\eta)\,d\eta$$

and $F(\theta)$ satisfies boundary and normality conditions. The boundary conditions for $F(\theta)$, as well as the evolution of the process when singularities exist, will be discussed in this section. We can rewrite $s(\theta)$ as

$$s(\theta)=e^{b(\theta)}$$

where

$$b(\theta)=-\int^{\theta}\frac{2\Phi(\eta)}{\Psi^2(\eta)}\,d\eta=\ln|\sin 2\theta|+(\tilde{\lambda}_2-\tilde{\lambda}_1)\,\beta(\theta)$$

In terms of the parameters A, B, and C,

$$\beta(\theta)=-\frac{1}{2}\int^{\cos 2\theta}\frac{dt}{At^2+Bt+C}$$

The last integral can be broken down into the following six cases:

1. $A, B, C \neq 0$.
2. $B=0$ and (i) $A, C \neq 0$, (ii) $A=0, C \neq 0$, (iii) $A \neq 0, C=0$.
3. $C=0$ and $A, B \neq 0$.
4. $A=0$ and $B, C \neq 0$.
5. $A=C=0$ and $B \neq 0$.
6. $A=B=C=0$.

Singularities in the $F(\theta)$ process are θ values satisfying either

$$\Psi^2(\theta)=0 \qquad \text{or} \quad \Phi(\theta)=\infty$$

These singularities can be classified according to Feller's scheme as entrance, exit, natural, or regular boundaries of a region of state space.[6] The following definitions, summarized by Karlin and Taylor,[7] are needed in order to classify the behavior of a stochastic process at a singular point θ_s and to determine the type of boundary present:

$$S(\theta_s, \theta] = \int_{\theta_s}^{\theta} s(\eta)\, d\eta, \qquad M(\theta_s, \theta] = \int_{\theta_s}^{\theta} m(\eta)\, d\eta$$

$$\Sigma(\theta_s) = \int_{\theta_s}^{\theta} S(\theta_s, \xi]\, dM(\xi) = \int_{\theta_s}^{\theta} \left\{ \int_{\eta}^{\theta} m(y)\, dy \right\} s(\eta)\, d\eta$$

$$N(\theta_s) = \int_{\theta_s}^{\theta} M(\theta_s, \xi]\, dS(\xi) = \int_{\theta_s}^{\theta} \left\{ \int_{\eta}^{\theta} s(y)\, dy \right\} m(\eta)\, d\eta$$

In the above definitions, $S(\theta_s, \theta]$ and $M(\theta_s, \theta]$ are the scale and speed measures, respectively. We will also use the notation $S(\theta)$ and $M(\theta)$ in which the explicit dependence on θ_s is dropped. The last two quantities measure the time it takes to reach the boundary of the state space starting from the interior, $\Sigma(\theta_s)$, and the time required to reach the interior beginning at the boundary, $N(\theta_s)$. In the above definitions, θ_s is a left boundary. Analogous definitions are employed when θ_s is a right boundary.

An *entrance boundary* cannot be reached from the interior of the state space but it is possible for the process to begin at such a point. The singular point is an entrance boundary if and only if

$$S(\theta_s, \theta] = \infty \qquad \text{and} \qquad N(\theta_s) < \infty$$

Once the process reaches an *exit boundary*, it is impossible to reenter the interior. The necessary and sufficient conditions for a singular point to be an exit are

$$M(\theta_s, \theta] < \infty \qquad \text{and} \qquad \Sigma(\theta_s) < \infty$$

An exit boundary within the state space implies that the process eventually exits a portion of the space and enters another region. The direction in which this boundary is traversed depends upon the sign of the drift term $\Phi(\theta_s)$ as follows:

$$\Phi(\theta_s) = \begin{cases} <0 & \text{left (backward) shunt} \\ >0 & \text{right (forward) shunt} \\ =0 & \text{trap} \end{cases}$$

A diffusion process can neither reach in finite mean time nor be started from a *natural* (*Feller*) *boundary*. A singular point is a natural boundary if and only if

$$N(\theta_s) = \infty \qquad \text{and} \qquad \Sigma(\theta_s) = \infty$$

Finally, a *regular boundary* allows the diffusion process to both enter and leave. The criteria for a regular boundary are

$$S(\theta_s, \theta] < \infty \qquad \text{and} \qquad M(\theta_s, \theta] < \infty$$

In this investigation, we will examine all possible singular cases. We can compute the function $F(\theta)$ and determine the behavior of the process at the singular points.

Employing the above criteria, it can be shown that for all six cases, the boundary points $\theta = (0, \pi/2)$ are entrance boundaries. Since $S(0) = -\infty$ and $S(\pi/2) = \infty$, in order for $F(\theta)$ to remain positive throughout the interval, it must be that $A_{st} = 0$ in Eq. (21), i.e., the zero-flux property. This leaves

$$F(\theta) = cm(\theta) \tag{22}$$

where c is the normalizing parameter. The expressions for $m(\theta)$ and the normalizing constant c for each case are given below.

Case 1.A, B, C $\neq 0$:

$$\Psi^2(\theta) = -\gamma \cos^2 2\theta - \tfrac{1}{2}(\alpha_{12} - \alpha_{21}) \cos 2\theta + \gamma + \tfrac{1}{2}(\alpha_{12} + \alpha_{21})$$

The only singularities for this case are the entrance boundaries $\theta = 0$ and

$\theta = \pi/2$ as shown in Fig. 2. We define the discriminant $\Delta = 4AC - B^2$ and consider the case $\Delta > 0$, $\Delta = 0$, and $\Delta < 0$.

For $\Delta > 0$

$$m(\theta) = \frac{\sin 2\theta}{\Psi^2(\theta)} \exp\left\{\frac{-(\tilde{\lambda}_2 - \tilde{\lambda}_1)}{\sqrt{\Delta}} \tan^{-1}\left(\frac{2A\cos 2\theta + B}{\sqrt{\Delta}}\right)\right\}$$

and

$$M(\theta) = \frac{1}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \exp\left\{\frac{-(\tilde{\lambda}_2 - \tilde{\lambda}_1)}{\sqrt{\Delta}} \tan^{-1}\left(\frac{2A\cos 2\theta + B}{\sqrt{\Delta}}\right)\right\}$$

For $\Delta = 0$

$$m(\theta) = \frac{\sin 2\theta}{\Psi^2(\theta)} \exp\left(\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{2A\cos 2\theta + B}\right)$$

and

$$M(\theta) = \frac{1}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \exp\left(\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{2A\cos 2\theta + B}\right)$$

For $\Delta < 0$

$$m(\theta) = \frac{\sin 2\theta}{\Psi^2(\theta)} \exp\left\{\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{\sqrt{-\Delta}} \tanh^{-1}\left(\frac{2A\cos 2\theta + B}{\sqrt{-\Delta}}\right)\right\}$$

and

$$M(\theta) = \frac{1}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \exp\left\{\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{\sqrt{-\Delta}} \tanh^{-1}\left(\frac{2A\cos 2\theta + B}{\sqrt{-\Delta}}\right)\right\}$$

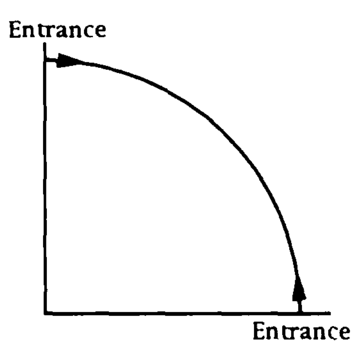

Fig. 2. Boundary behavior for singular Cases 1, 2i ($AC > 0$), 2ii, and 4.

In all of the above cases for Δ, the normalizing constant c is

$$c = [M(\pi/2) - M(0)]^{-1}$$

where the appropriate $M(\theta)$ must be used.

Case 2i. $B = 0$ and A, $C \neq 0$:

$$\Psi^2(\theta) = -\gamma \cos^2 2\theta + \gamma + \alpha$$

$B = 0$ implies $\alpha_{12} = \alpha_{21} = \alpha$. As in Case 1, the singularities at $\theta = (0, \pi/2)$ are entrance boundaries. There is, however, an additional singularity in this case at

$$\theta = \theta_s = \frac{1}{2} \cos^{-1} \left(\frac{-C}{A} \right)^{1/2}$$

which is valid only when $AC < 0$. The function $m(\theta)$ is given by

$$m(\theta) = \begin{cases} \dfrac{\sin 2\theta}{\Psi^2(\theta)} \exp \left\{ \dfrac{-(\tilde{\lambda}_2 - \tilde{\lambda}_1)}{2(AC)^{1/2}} \tan^{-1} \left(\dfrac{(AC)^{1/2}}{C} \cos 2\theta \right) \right\}, & AC > 0 \\ \dfrac{\sin 2\theta}{\Psi^2(\theta)} \exp \left\{ \dfrac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{2(-AC)^{1/2}} \tanh^{-1} \left(\dfrac{(-AC)^{1/2}}{C} \cos 2\theta \right) \right\}, & AC < 0 \end{cases}$$

In order to determine the normalizing constant c, we must consider the two cases $AC > 0$ and $AC < 0$ separately. For $AC > 0$, since no singularity exists in the open interval $(0, \pi/2)$, c is simply

$$c = \tfrac{1}{2} (\tilde{\lambda}_2 - \tilde{\lambda}_1) \operatorname{csch} \left\{ \frac{(\tilde{\lambda}_2 - \tilde{\lambda}_1)}{2(AC)^{1/2}} \tan^{-1} \left(\frac{(AC)^{1/2}}{C} \right) \right\}$$

When $AC < 0$, it can be shown that the point $\theta = \theta_s$ is a left or right shunt, depending on the sign of $\Phi(\theta_s)$, where

$$\Phi(\theta_s) = \frac{1}{2} (\tilde{\lambda}_2 - \tilde{\lambda}_1) \left(1 + \frac{C}{A} \right)^{1/2}$$

For $\tilde{\lambda}_1 > \tilde{\lambda}_2$

$$F(\theta) = \begin{cases} cm(\theta), & \theta \in (0, \theta_s) \\ 0, & 0 \in (\theta_s, \pi/2) \end{cases}$$

In this case, the point $\theta = \theta_s$ is a left shunt. A process starting from a point in the region $\theta \in (\theta_s, \pi/2)$ will eventually leave this region and the shunted over to the portion of state space bounded by $\theta \in (0, \theta_s)$. This leads to a buildup of probability in the region $\theta \in (0, \theta_s)$.

For $\tilde{\lambda}_1 < \lambda_2$

$$F(\theta) = \begin{cases} 0, & \theta \in (0, \theta_s) \\ cm(\theta), & 0 \in (\theta_s, \pi/2) \end{cases}$$

Here, $\theta = \theta_s$ is a right shunt. A process starting from a point in the region $\theta \in (0, \theta_s)$ will eventually be shunted over to the region bounded by $\theta \in (\theta_s, \pi/2)$ and the probability accumulates in this region. In both cases,

$$c = |\tilde{\lambda}_2 - \tilde{\lambda}_1| \exp\left(\frac{|\tilde{\lambda}_2 - \tilde{\lambda}_1|}{2A}\right)$$

The singular behavior for this case is summarized in Fig. 3.

Case 2ii. $B = 0$ and $A = 0$, $C \neq 0$:

$$\Psi^2(\theta) = \alpha$$

As in Case 1, the only singularities are at $\theta = (\theta, \pi/2)$ (see Fig. 2) and $m(\theta)$ and c are given by

$$m(\theta) = \frac{\sin 2\theta}{\alpha} \exp\left\{\frac{-(\tilde{\lambda}_2 - \tilde{\lambda}_1)}{2\alpha} \cos 2\theta\right\}$$

and

$$c = \frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{2} \operatorname{csch}\left(\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{2\alpha}\right)$$

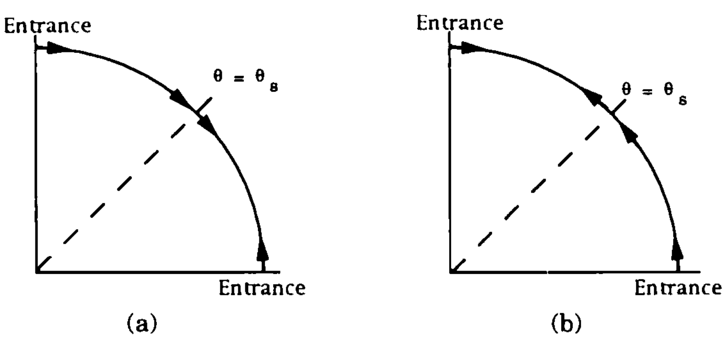

Fig. 3. Boundary behavior for singular Case 2i ($AC < 0$) and Case 2iii (a) $\tilde{\lambda}_1 > \tilde{\lambda}_2$, (b) $\tilde{\lambda}_1 < \tilde{\lambda}_2$.

Case 2iii. $B=0$ and $A\neq 0$, $C=0$:

$$\Psi^2(\theta)=\alpha\cos^2 2\theta$$

In addition to the singularities at the boundaries $\theta=(0,\pi/2)$, there is also a singular point at $\theta=\pi/4$. As in Case 2i, the sign of $\Phi(\theta_s)$ determines the behavior of the process at this point. In this case

$$\Phi(\theta_s)=\tfrac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)$$

For $\tilde{\lambda}_1>\tilde{\lambda}_2$

$$F(\theta)=\begin{cases} cm(\theta), & \theta\in(0,\pi/4) \\ 0, & \theta\in(\pi/4,\pi/2) \end{cases}$$

Thus, the point $\theta=\pi/4$ is a left shunt. This leads to a buildup of probability in the region $\theta\in(0,\pi/4)$.

For $\tilde{\lambda}_1<\tilde{\lambda}_2$

$$F(\theta)=\begin{cases} 0, & \theta\in(0,\pi/4) \\ cm(\theta), & \theta\in(\pi/4,\pi/2) \end{cases}$$

Here, $\theta=\pi/4$ is a right shunt and the probability accumulates in the region $\theta\in(\pi/4,\pi/2)$. In both of the above cases,

$$m(\theta)=\frac{\sin 2\theta}{\Psi^2(\theta)}\exp\left(\frac{\tilde{\lambda}_2-\tilde{\lambda}_1}{2\alpha}\sec 2\theta\right)$$

and

$$c=|\tilde{\lambda}_2-\tilde{\lambda}_1|\exp\left(\frac{|\tilde{\lambda}_2-\tilde{\lambda}_1|}{2\alpha}\right)$$

Figure 3 summarizes the behavior of the diffusion process for this case.

Case 3. $C=0$ and $A, B\neq 0$:

$$\Psi^2(\theta)=A\cos^2 2\theta+B\cos 2\theta$$

In this case, we have singular points at

$$\theta=\frac{\pi}{4}\qquad\text{and}\qquad\theta=\theta_s=\frac{1}{2}\cos^{-1}\left(\frac{-B}{A}\right)=\frac{1}{2}\cos^{-1}\left(\frac{\alpha^-}{\alpha^+}\right)$$

as well as at the boundaries $\theta=(0,\pi/2)$.

For $\tilde{\lambda}_1 > \tilde{\lambda}_2$ and $B < 0$

$$F(\theta) = c_1 \begin{cases} m(\theta), & \theta \in (0, \theta_s) \\ \delta(\theta - \pi/4), & \theta \in (\theta_s, \pi/2) \end{cases}$$

By Feller's scheme, the point $\theta = \theta_s$ is an entrance and $\theta = \pi/4$ is an exit boundary.

For $\tilde{\lambda}_1 > \tilde{\lambda}_2$ and $B > 0$

$$F(\theta) = c_1 \begin{cases} m(\theta), & \theta \in (0, \pi/4) \\ \delta(\theta - \theta_s), & \theta \in (\pi/4, \pi/2) \end{cases}$$

The singularity at $\theta = \theta_s$ is an exit and $\theta = \pi/4$ is an entrance boundary.

For $\tilde{\lambda}_1 < \tilde{\lambda}_2$ and $B < 0$

$$F(\theta) = c_2 \begin{cases} \delta(\theta - \theta_s), & \theta \in (0, \pi/4) \\ m(\theta), & \theta \in (\pi/4, \pi/2) \end{cases}$$

The point $\theta = \theta_s$ is an exit and $\theta = \pi/4$ is an entrance.

For $\tilde{\lambda}_1 < \tilde{\lambda}_2$ and $B > 0$

$$F(\theta) = c_2 \begin{cases} \delta(\theta - \pi/4), & \theta \in (0, \theta_s) \\ m(\theta), & \theta \in (\theta_s, \pi/2) \end{cases}$$

The singular point $\theta = \theta_s$ is an entrance boundary and $\theta = \pi/4$ is an exit. The boundary behavior for this case is depicted in Fig. 4. In all cases,

$$m(\theta) = \frac{\sin 2\theta}{\Psi^2(\theta)} \left| -\alpha^+ + \alpha^- \sec 2\theta \right|^{(\tilde{\lambda}_1 - \tilde{\lambda}_2)/\alpha^-}$$

and

$$c_1 = \frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{\tilde{\lambda}_2 - \tilde{\lambda}_1 - (2\alpha_{21})^{(\tilde{\lambda}_1 - \tilde{\lambda}_2)/\alpha^-}}, \qquad c_2 = \frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{\tilde{\lambda}_2 - \tilde{\lambda}_1 + (2\alpha_{12})^{(\tilde{\lambda}_1 - \tilde{\lambda}_2)/\alpha^-}}$$

Case 4. $A = 0$ and B, $C \neq 0$:

$$\Psi^2(\theta) = \tfrac{1}{2}\alpha^+ - \tfrac{1}{2}\alpha^- \cos 2\theta$$

The only singular points for this case are at the boundaries $\theta = (0, \pi/2)$ (see Fig. 2) since the singular point defined by

$$\theta_s = \frac{1}{2} \cos^{-1}\left(\frac{\alpha^+}{\alpha^-}\right)$$

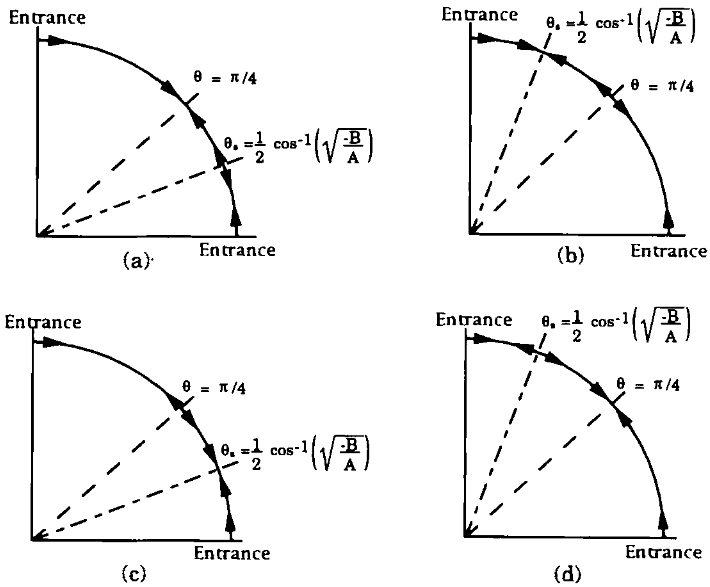

Fig. 4. Boundary behavior for singular Case 3. (a) $\tilde{\lambda}_1 > \tilde{\lambda}_2$, $B<0$, (b) $\tilde{\lambda}_1 > \tilde{\lambda}_2$, $B>0$, (c) $\tilde{\lambda}_1 < \tilde{\lambda}_2$, $B<0$, (d) $\tilde{\lambda}_1 < \tilde{\lambda}_2$, $B>0$.

can be shown to coincide with either $\theta = 0$ or $\theta = \pi/2$ due to the condition

$$0 \leqslant \left|\frac{\alpha^+}{\alpha^-}\right| \leqslant 1$$

In this case,

$$m(\theta) = \frac{\sin 2\theta}{\Psi^2(\theta)} \left|-\alpha^+ + \alpha^- \cos 2\theta\right|^{(\tilde{\lambda}_2 - \tilde{\lambda}_1)/\alpha^-}$$

and

$$c = \frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{(2\alpha_{12})^{(\tilde{\lambda}_2 - \tilde{\lambda}_1)/\alpha^-} - (2\alpha_{21})^{(\tilde{\lambda}_2 - \tilde{\lambda}_1)/\alpha^-}}$$

Case 5. $A = C = 0$ and $B \neq 0$. Satisfying $A = C = 0$ requires the following conditions:

$$\gamma = 0 \qquad \text{and} \qquad \alpha^+ = 0$$

Since $\alpha^+ = \alpha_{12} + \alpha_{21}$, where α_{12} and α_{21} are nonnegative quantities, the second requirement implies $\alpha_{12}=\alpha_{21}=0$. For $B\neq 0$, the following condition must hold:

$$\alpha^{-}\neq 0$$

These cannot be satisfied simultaneously. Thus, the Case 5 singularity is not possible.

Case 6. $A=B=C=0$:

$$\Psi^2(\theta)=0 \qquad \forall\theta\in(0,\pi/2)$$

This corresponds to the case of two uncoupled oscillators. In this case, the diffusion process is singular for all values of θ and a function $F(\theta)$ satisfying the normality condition over the interval $(0,\pi/2)$ cannot be found. However, as stated previously, the Lyapunov exponents for this situation are λ_1 and λ_2. The maximal Lyapunov exponent is simply the greater of the two.

4. MAXIMAL LYAPUNOV EXPONENTS

The maximal Lyapunov exponent given by Eq. (12) can now be calculated. Letting

$$J(\theta)=\Psi^2(\theta)+\tfrac{1}{2}(\tilde{\lambda}_1-\tilde{\lambda}_2)\cos 2\theta$$

and

$$\tilde{c}=\tfrac{1}{2}(\tilde{\lambda}_1+\tilde{\lambda}_2)-\tfrac{1}{2}(\gamma_1+\gamma_2)+\mu$$

and considering terms up to $O(\varepsilon^2)$ only yields the following for the expansion in λ^{ε}:

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\int_0^{\pi/2}J(\theta)\,F(\theta)\,d\theta+\tilde{c}\int_0^{\pi/2}F(\theta)\,d\theta\right\} \tag{23}$$

Integration by parts and substitution of the appropriate $F(\theta)$ yields the maximal Lyapunov exponent for each of the singular cases.

Case 1. A, B, $C\neq 0$. For all cases ($\Delta>0$, $\Delta=0$, $\Delta<0$), using the appropriate $M(\theta)$,

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\frac{[M(\pi/2)+M(0)]}{[M(\pi/2)-M(0)]}+\tilde{c}\right\}$$

Case 2i. $B=0$ and A, $C \neq 0$. For $AC>0$:

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\coth\left[\frac{(\tilde{\lambda}_2-\tilde{\lambda}_1)}{(AC)^{1/2}}\tan^{-1}\left(\frac{(AC)^{1/2}}{C}\right)\right]+\tilde{c}\right\}$$

For $AC<0$, recall we have the additional singularity at $\theta=\theta_s$. The maximal Lyapunov exponent is given by

$$\lambda^{\varepsilon}=\varepsilon^2\left(\frac{|\tilde{\lambda}_2-\tilde{\lambda}_1|}{2}\exp\left\{\frac{|\tilde{\lambda}_2-\tilde{\lambda}_1|}{2}\left[1-\frac{1}{(-AC)^{1/2}}\tanh^{-1}\left(\frac{(-AC)^{1/2}}{C}\right)\right]\right\}\right.$$
$$\left.+\frac{|\tilde{\lambda}_2-\tilde{\lambda}_1|}{2}(\tilde{\lambda}_1-\tilde{\lambda}_2)\frac{(-AC)^{1/2}}{A}\exp\left(\frac{|\tilde{\lambda}_2-\tilde{\lambda}_1|}{2A}\right)+\tilde{c}\right)$$

Case 2ii. $B=0$ and $A=0$, $C\neq 0$:

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\coth\left(\frac{\tilde{\lambda}_2-\tilde{\lambda}_1}{2\alpha}\right)+\tilde{c}\right\}$$

Case 2iii. $B=0$ and $A\neq 0$, $C=0$:

$$\lambda^{\varepsilon}=\varepsilon^2\left(\tfrac{1}{2}|\tilde{\lambda}_2-\tilde{\lambda}_1|+\tilde{c}\right)$$

Case 3. $C=0$ and A, $B\neq 0$. For $\tilde{\lambda}_1>\tilde{\lambda}_2$ and $B<0$

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\left[\frac{(2\alpha_{21})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}}{\tilde{\lambda}_2-\tilde{\lambda}_1-(2\alpha_{21})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}}\right]+\tilde{c}\right\}$$

For $\tilde{\lambda}_1>\tilde{\lambda}_2$ and $B>0$

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\left[\frac{(2\alpha_{21})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}-(\alpha^-/\alpha^+)(\tilde{\lambda}_2-\tilde{\lambda}_1)}{\tilde{\lambda}_2-\tilde{\lambda}_1-(2\alpha_{21})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}}\right]+\tilde{c}\right\}$$

For $\tilde{\lambda}_1<\tilde{\lambda}_2$ and $B<0$

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\left[\frac{(2\alpha_{12})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}-(\alpha^-/\alpha^+)(\tilde{\lambda}_2-\tilde{\lambda}_1)}{\tilde{\lambda}_2-\tilde{\lambda}_1+(2\alpha_{12})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}}\right]+\tilde{c}\right\}$$

For $\tilde{\lambda}_1<\tilde{\lambda}_2$ and $B>0$

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\left[\frac{(2\alpha_{12})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}}{\tilde{\lambda}_2-\tilde{\lambda}_1+(2\alpha_{12})^{(\tilde{\lambda}_1-\tilde{\lambda}_2)/\alpha^-}}\right]+\tilde{c}\right\}$$

Case 4. $A=0$ and B, $C\neq 0$:

$$\lambda^{\varepsilon}=\varepsilon^2\left\{\frac{1}{2}(\tilde{\lambda}_2-\tilde{\lambda}_1)\left[\frac{(2\alpha_{12})^{(\tilde{\lambda}_2-\tilde{\lambda}_1)/\alpha^-}+(2\alpha_{21})^{(\tilde{\lambda}_2-\tilde{\lambda}_1)/\alpha^-}}{(2\alpha_{12})^{(\tilde{\lambda}_2-\tilde{\lambda}_1)/\alpha^-}-(2\alpha_{21})^{(\tilde{\lambda}_2-\tilde{\lambda}_1)/\alpha^-}}\right]+\tilde{c}\right\}$$

Case 6. $A=B=C=0$:

$$\lambda^{\varepsilon}=\varepsilon^2\max(\lambda_1,\lambda_2)$$

It is important to note that when the conditions of Eq. (3) are imposed, $\gamma_1=\gamma_2=0$, the maximal Lyapunov exponent given by Eq. (23) is identical to that obtained via the method of averaging.[17]

5. ROTATION NUMBERS

The Multiplicative Ergodic Theorem for rotation numbers is given in a recent paper by Arnold and San Martin[5] in which a general method is given for calculating the rotation numbers ρ_{ij} of the canonical planes $p_{ij}=\text{span}(E_i, E_j)$, where E_i and E_j are the Oseledec spaces described earlier. The rotation number of any other plane will pick the value ρ_{ij} whenever this plane has p_{ij} as the strongest component. In the current calculations, three angles have been introduced. Thus, by definition, there would be a rotation number associated with each angle giving the exponential rotation rate. Since these angles are not defined with respect to canonical bases, the rotation numbers cannot be readily related to those given in ref. 5. However, it is clear that if the plane of x_1–x_2 corresponds to any of the p_{ij}, then one can relate α_1 to ρ_{ij}. A similar relation exists for α_2. The relationship between the results of Arnold and San Martin and those presented here must be further investigated.

The rotation number, the stochastic analog of the imaginary part of the eigenvalue for the linear system, is calculated in this section. In terms of the invariant measure p^{ε}, the rotation numbers are given as

$$\alpha_i^{\varepsilon}=\langle H_i^{\varepsilon}, p^{\varepsilon}\rangle \tag{24}$$

where

$$H_i^{\varepsilon}(\phi_1,\phi_2,\theta,\xi)=\omega_i+\varepsilon f(\xi)\,h_i(\phi_1,\phi_2,\theta)+\varepsilon^2\tilde{h}_i(\phi_1,\phi_2,\theta)$$

Again, for A_1 as given, $\tilde{h}_i=0$.

The rotation number given by Eq. (24) can be rewritten as

$$\begin{aligned}\alpha_i^{\varepsilon}=\langle\omega_i, p^0\rangle&+\varepsilon[\langle\omega_i, p^1\rangle+\langle f(\xi)\,h_i, p^0\rangle]\\&+\varepsilon^2[\langle\omega_i, p^2\rangle+\langle f(\xi)\,h_i, p^1\rangle+\langle\tilde{h}_i, p^0\rangle]\end{aligned} \tag{25}$$

where ω_i is a constant and p^ε is scaled such that p^1 and p^2 have mean zero. The last term is zero due to the structure of the A_1 matrix. Due to the periodic boundary conditions on ϕ_1 and ϕ_2, as well as the zero mean assumption on $f(\xi)$, we have

$$\langle f(\xi)\, h_i,\, p^0 \rangle = 0$$

Hence, Eq. (25) reduces to

$$\alpha_i^\varepsilon = \omega_i + \varepsilon^3 \langle f(\xi)\, h_i,\, p^1 \rangle \tag{26}$$

Making use of the definitions

$$\begin{aligned}
\hat{\alpha}_{12} &= \tfrac{1}{8}\{[(H_{12}^-)^2 + (J_{12}^+)^2)]\, \Gamma(\Omega^-) + [(H_{12}^+)^2 + (J_{12}^-)^2]\, \Gamma(\Omega^+)\} \\
\hat{\alpha}_{21} &= \tfrac{1}{8}\{[(H_{21}^-)^2 + (J_{21}^+)^2]\, \Gamma(\Omega^-) - [(H_{21}^+)^2 + (J_{21}^-)^2]\, \Gamma(\Omega^+)\} \\
\hat{\beta}_i &= \tfrac{1}{8}[(H_{ii}^+)^2 + (J_{ii}^-)^2]\, \Gamma(2\omega_i) \\
\hat{\mu}_1 &= \tfrac{1}{8}[(-H_{12}^- H_{21}^- + J_{12}^+ J_{21}^+)\, \Gamma(\Omega^-)] \\
\hat{\mu}_2 &= \tfrac{1}{8}[(H_{12}^+ H_{21}^+ + J_{12}^- J_{21}^-)\, \Gamma(\Omega^+) \\
\hat{\gamma}_1 &= \tfrac{1}{4}(H_{21}^- J_{12}^+ + H_{12}^- J_{21}^+)\, S(\Omega^-) \\
\hat{\gamma}_2 &= \tfrac{1}{4}(H_{21}^+ J_{12}^- - H_{12}^+ J_{21}^-)\, S(\Omega^+)
\end{aligned}$$

we can write the rotation numbers as

$$\alpha_1 = \omega_1 - \varepsilon^2\{\hat{\beta}_1 + \tfrac{1}{2}(\hat{\gamma}_2 - \hat{\gamma}_1) + (\hat{\mu}_1 + \hat{\mu}_2) + \tfrac{1}{2}\hat{\alpha}_{12} \lim_{\theta \to \pi/2} [\sec(\theta)\, F(\theta)]\} \tag{27}$$

$$\alpha_1 = \omega_2 - \varepsilon^2\{\hat{\beta}_2 - \tfrac{1}{2}(\hat{\gamma}_2 + \hat{\gamma}_1) - (\hat{\mu}_1 - \hat{\mu}_2) + \tfrac{1}{2}\hat{\alpha}_{21} \lim_{\theta \to 0} [\csc(\theta)\, F(\theta)]\} \tag{28}$$

By substituting the appropriate $F(\theta)$ into the above expressions, we can find the rotation numbers explicitly for each of the possible singular cases.

6. CONCLUSIONS

In this paper, an asymptotic expansion for the maximal Lyapunov exponent, the exponential growth rate of solutions to a linear stochastic system, and the rotation numbers for a general four-dimensional dynamical system driven by a small-intensity real noise process were constructed. Stability boundaries, defined as the points at which the maximal Lyapunov exponent becomes zero, can then be obtained provided the natural frequencies are noncommensurable and the infinitesimal generator associated with the noise process has an isolated simple zero eigenvalue. This last assumption was made to make the solution tractable. The advantage of this method over the method of stochastic averaging is the applicability of the perturbation approach to problems in the form of Eq. (1) without imposing any conditions on the form of the B matrix. For systems in which these conditions hold, the expressions for the maximal Lyapunov exponent and the rotation numbers calculated here reduce to those presented by Sri Namachchivaya and Talwar.[17]

It is worth pointing out that there are some other quantities that may be obtained asymptotically employing the current technique. On can use Liouville's theorem to calculate the sum of the Lyapunov exponents as the expected value of the trace of the complete linear coefficient matrix, i.e., $E(\text{Trace}[A_0 - \varepsilon^2 A_1 + \varepsilon B f(\xi)]) = \sum_{i=1}^{4} \lambda_i$. Considering the definition of the matrices A_0 and A_1 and the zero mean assumption on the noise process, the sum of the Lyapunov exponents is simply $2\varepsilon^2(\delta_1 + \delta_2)$. In two-dimensional systems, the trace and the top Lyapunov exponent completely describe the spectrum. However, the authors are not aware of methods of describing the complete spectrum for systems with dimension greater than three. It should also be noted that the smallest Lyapunov exponent can be obtained by following the same procedure given here with time reversed. In this case, attention must be paid to the various generators describing the noise process.

APPENDIX

The explicit expressions for q_i, $\tilde{q}_i$, and h_i $(i=1, 2)$ used in Eqs. (10) are

$$
\begin{aligned}
q_1(\phi_1, \phi_2, \theta) = {} & \tfrac{1}{2}(J_{11}^{+} - J_{11}^{-} \cos 2\phi_1 - H_{11}^{+} \sin 2\phi_1) \cos^2 \theta \\
& + \tfrac{1}{2}(J_{22}^{+} - J_{22}^{-} \cos 2\phi_2 - H_{22}^{+} \sin 2\phi_2) \sin^2 \theta \\
& + \tfrac{1}{2}[(J_{12}^{+} + J_{21}^{+}) \cos(\phi_1 - \phi_2) \\
& - (J_{12}^{-} + J_{21}^{-}) \cos(\phi_1 + \phi_2) \\
& + (H_{12}^{-} - H_{21}^{-}) \sin(\phi_1 - \phi_2) \\
& - (H_{12}^{+} + H_{21}^{+}) \sin(\phi_1 + \phi_2)] \cos \theta \sin \theta \\
\tilde{q}_1(\theta) = {} & -\delta_1 \cos^2 \theta - \delta_2 \sin^2 \theta
\end{aligned}
$$

$$
\begin{aligned}
h_1(\phi_1, \phi_2, \theta) = {} & \tfrac{1}{2}(H_{11}^{-} - H_{11}^{+} \cos 2\phi_1 + J_{11}^{-} \sin 2\phi_1) \\
& + \tfrac{1}{2}[H_{12}^{-} \cos(\phi_1 - \phi_2) - H_{12}^{+} \cos(\phi_1 + \phi_2) \\
& - J_{12}^{+} \sin(\phi_1 - \phi_2) + J_{12}^{-} \sin(\phi_1 + \phi_2)] \tan \theta
\end{aligned}
$$

$$h_2(\phi_1, \phi_2, \theta) = \tfrac{1}{2}(H_{22}^- - H_{22}^+ \cos 2\phi_2 + J_{22}^- \sin 2\phi_2)$$
$$+ \tfrac{1}{2}[H_{21}^- \cos(\phi_1 - \phi_2) - H_{21}^+ \cos(\phi_1 + \phi_2)$$
$$+ J_{21}^+ \sin(\phi_1 - \phi_2) + J_{21}^- \sin(\phi_1 + \phi_2)] \cot \theta$$
$$q_2(\phi_1, \phi_2, \theta) = \tfrac{1}{2}[J_{22}^+ - J_{11}^+ + J_{11}^- \cos 2\phi_1 + H_{11}^+ \sin 2\phi_1$$
$$- J_{22}^- \cos 2\phi_2 - H_{22}^+ \sin 2\phi_2] \cos \theta \sin \theta$$
$$+ \tfrac{1}{2}[-J_{12}^+ \cos(\phi_1 - \phi_2) + J_{12}^- \cos(\phi_1 + \phi_2)$$
$$- H_{12}^- \sin(\phi_1 - \phi_2) + H_{12}^+ \sin(\phi_1 + \phi_2)] \sin^2 \theta$$
$$+ \tfrac{1}{2}[J_{21}^+ \cos(\phi_1 - \phi_2) - J_{21}^- \cos(\phi_1 + \phi_2)$$
$$- H_{21}^- \sin(\phi_1 - \phi_2) - H_{21}^+ \sin(\phi_1 + \phi_2)] \cos^2 \theta$$
$$\tilde{q}_2(\theta) = (\delta_1 - \delta_2) \sin \theta \cos \theta$$

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